

# N=2 HETEROTIC STRINGY COSMIC STRINGS<sup>\*</sup>

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## Abstract

We construct solitonic string solutions of N=2 four-dimensional heterotic models of rank three, four and five. These finite energy configurations have constant dilaton while the moduli fields vary over space-time with jumps at the location of the string cores consistent with the T-duality groups  $SL(2, \mathbf{Z})$ ,  $SL(2, \mathbf{Z}) \times SL(2, \mathbf{Z})$  and  $Sp(4, \mathbf{Z})$ . The solutions are expressed in terms of modular forms of the T-duality group. They break half of the supersymmetries and the vacuum contain a certain number of solitonic strings in order the singularities to be resolved in a Ricci flat way.

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## 1. Introduction

Theories with extended supersymmetries reveal a rich dynamical structure [1]–[4]. An important feature of these theories is the existence of BPS states which break half of the supersymmetries. They are of particular importance in determine the dynamics of the theory and their semi-classical analysis in some cases is enough to determine also their strong coupling behaviour. On the other hand, they are necessary for the consistency of the theory. For example, BPS states which carry Ramond–Ramond charge are the D–branes [5] of type II theory where the type I string may have its ends. They also play a central role in establishing the various string/string dualities, in M– and F–theory, in various string compactifications and so on [4],[6]–[9]. It seems that all the recent developments indicate that the understanding of the structure of the BPS states will be a central issue in unrevealing the secrets of string theory.

In this paper, we will deal with four–dimensional supersymmetric N=2 models and we will construct string–like configurations which satisfy a Bogomol’nyi bound and break half of the space–time supersymmetries. N=2 supersymmetry in four dimensions can be obtained by compactifying type II strings on a Calabi–Yau threefolds with Betti numbers  $h_{11}$  and  $h_{12}$ . The number  $h_{11}$  in type IIA theory for example, gives the number of vector multiplets and combined with the graviphoton, the rank of the gauge group turns out to be  $h_{11} + 1$ . On the other hand, the dilaton belongs to a hypermultiplet and the total number of hypermultiplets is  $h_{12} + 1$ . The tree level prepotential is exact in the full quantum theory since the dilaton belongs to a hypermultiplet and the mirror symmetry made possible its exact computation [11]–[13]. There exist in general logarithmic singularities near the conifold locus in the moduli space of the CY threefold [13]. The problem of these conifolds singularities resolved by Strominger who proposed that hypermultiplets corresponding to charged black holes become massless near the conifold locus [10].

The compactification of the heterotic string on  $K_3 \times T^2$  also give rise to N=2 supersymmetry in four dimensions with  $n_V + 1$  vector multiplets, including the graviphoton, and  $n_H$  hyper multiplets. The rank of the gauge group is  $n_V + 2$  in this case since the dilaton belongs to a vector multiplet now. Supersymmetry requires the moduli space of the vector multiplets to be a special Kähler manifold  $\mathcal{K}_{n_V}$  while the moduli space of the

hypermultiplets is a quaternionic manifold  $\mathcal{Q}_{n_H}$  [14],[15]. In fact, the vector multiplets parametrize the coset  $\frac{SU(1,1)}{U(1)} \times \frac{O(2,n_V)}{O(2) \times O(n_V)}$  where the first factor is the moduli space of the S–dilaton. The classical T–duality group is  $O(2, n_V; \mathbf{Z})$  which, however, is modified by quantum corrections [17].

Here, we will consider string-like soliton solutions to the low-energy four-dimensional effective action which break half of the supersymmetries. We discuss first an N=4 model with dilaton  $S$  and two additional moduli  $T$  and  $U$ . The major difficulty in constructing solitonic string configurations comes from the fact that, without any extra assumption, all of them have infinite energy per unit length which leads to inconsistencies in solving the field equations [18]. However, one may construct finite energy solutions by employing the S– and T–duality groups and restricting the moduli fields to their fundamental domain. The moduli space has then finite volume and the energy turns out to be proportional to it.

In the case of N=2 heterotic strings in four–dimensions we explicitly discuss the rank three, four and five models with T–duality groups  $SL(2, \mathbf{Z})$ ,  $SL(2, \mathbf{Z})_T \times SL(2, \mathbf{Z})_U$  and  $Sp(4, \mathbf{Z})$ , respectively. These models have also been discussed in connection with heterotic/type IIA string duality [21]–[23]. Here again, the string configurations have infinite energy and finite energy solutions can only be constructed if one allows the moduli fields to have discontinuous jumps as they go around the string as long as these jumps have been done by an element of the T-duality group. In other words, solitonic solutions exist only if the fields are restricted to the fundamental domain of the T–duality group. One should recall at this point the stringy cosmic string of Greene et. al. where the  $SL(2, \mathbf{Z})$  T-duality group of a torus compactifications was been employed in order finite energy solutions to be constructed for a single moduli [18]. It should be noted that we fix the value of the S–dilaton since the S–duality group is lacking in the N=2 case.

String-like configurations which break half of the supersymmetries have previously been exploited as well. In [19], for example, a multi-string solution of the three–form ten–dimensional supergravity coupled to a string  $\sigma$ –model source has been constructed. This solutions was shown to satisfy a Bogomol’nyi bound and to break half of the space–time supersymmetries. Strictly speaking, it is not a genuine soliton since requires the presence of an “electrically” charged source due to a singularity at the location of the string. However,

it can be interpreted as a soliton of the fivebrane theory. Other genuine solitonic string solutions have also been constructed [20] and a review of the subject can be found in [3].

In the next chapter, we present the general setting of our constructions by discussing the solitonic string solutions in  $\sigma$ -models coupled to gravity. We also recall the Greene et al. solutions and we find stringy cosmic string solutions in the N=4 heterotic theory with three moduli  $S, T, U$  in four dimensions. In chapter 3 we construct solitonic string solutions of the rank three, four and five  $S - T$ ,  $S - T - U$  and  $S - T - U - V$  models. In chapter 4 we discuss some issues of our solutions and finally, in an appendix we present some properties of the Siegel modular group  $Sp(4, \mathbf{Z})$ .

## 2. Solitonic string solutions

### 2.1 General setting

The number of supersymmetries after compactification down to four dimensions is determined by the number of covariantly constant spinors in the internal six-dimensional space. For example, a Calabi–Yau compactification give rise to N=1 (2) supersymmetry while a  $K_3 \times T^2$  or a  $T^6$  compactification give N=2 (4) and N=4 (8) supersymmetry in four dimensions for heterotic (type II) strings. The form of the moduli space is then restricted by supersymmetry and for the N=2 case turns out to be  $\mathcal{M} = \mathcal{K} \times \mathcal{Q}$  where  $\mathcal{K}$  is a Kähler manifold for the moduli space of the vector multiplets and  $\mathcal{Q}$  is quaternionic for the hyper multiplets.

Let us consider the universal part of the effective action of the N=2 four-dimensional heterotic string which describes the dynamics of the graviton and the scalar components of the vector multiplets. The moduli space of the latter is a special Kähler manifold  $\mathcal{K}$  with local coordinates  $(w^i, \bar{w}^{\bar{i}}; i = 1, \dots, \dim_{\mathbf{C}} \mathcal{K})$ . The metric on  $\mathcal{K}$  is  $h_{i\bar{j}} = \partial_i \partial_{\bar{j}} K(w, \bar{w})$  where  $K(w, \bar{w})$  is the Kähler potential. The bosonic part of the one-loop corrected effective action up to first order in  $\alpha'$ -expansion is

$$\begin{aligned}
 I = & \int d^4x \sqrt{-g} \left( \frac{1}{2} R - h_{i\bar{j}} \partial_\mu w^i \partial^\mu \bar{w}^{\bar{j}} + \frac{1}{8} S_2 R_{GB}^2 + \frac{1}{8} S_1 R R^* \right. \\
 & \left. + \Delta(w^i, \bar{w}^{\bar{i}}) R_{GB}^2 + \Theta(w^i, \bar{w}^{\bar{i}}) R R^* \right), \tag{2.1}
 \end{aligned}$$

where  $\Delta(w^i, \bar{w}^i)$ ,  $\Theta(w^i, \bar{w}^i)$  are the moduli-dependent one-loop corrections [24],  $S_1, S_2$  are the real and imaginary parts of the S-dilaton and  $R_{GB}^2, RR^*$  are the CP-even Gauss-Bonnet combination and the CP-odd term defined by

$$\begin{aligned} R_{GB}^2 &= R_{\kappa\lambda\mu\nu} R^{\kappa\lambda\mu\nu} - 4R_{\mu\nu} R^{\mu\nu} + R^2, \\ RR^* &= \epsilon^{\kappa\lambda\mu\nu} R_{\kappa\lambda}{}^{\alpha\beta} R_{\mu\nu\alpha\beta}. \end{aligned} \quad (2.2)$$

We are looking for solitonic string-like solutions of the form

$$ds^2 = -dt^2 + dx_3^2 + e^{\rho(z, \bar{z})} dz d\bar{z}, \quad (2.3)$$

where the complex coordinates  $(z, \bar{z})$  parametrize the plane transverse to the string which is extended in the  $x_3$ -directions and the complex moduli are  $w^i = w^i(z, \bar{z})$ . With this form of the metric, both the Gauss-Bonnet and the CP-odd terms vanish and it is consistent with the equations of motions to ignore them. Thus, the effective actions takes the form of a  $\sigma$ -model coupled to gravity

$$I = \int d^4x \sqrt{-g} \left( \frac{1}{2} R - h_{i\bar{j}} \partial_\mu w^i \partial^\mu \bar{w}^j \right), \quad (2.4)$$

and the equations of motions are then

$$R_{\mu\nu} = h_{i\bar{j}} \partial_\nu w^i \partial_\mu \bar{w}^j + h_{i\bar{j}} \partial_\mu w^i \partial_\nu \bar{w}^j, \quad (2.5)$$

$$0 = \frac{1}{\sqrt{-g}} \partial_\mu \left( g^{\mu\nu} \sqrt{-g} \partial_\nu h_{i\bar{j}} w^i \right) - h_{i\bar{k}, \bar{j}} \partial_\mu w^i \partial^\mu \bar{w}^j, \quad (2.6)$$

where  $h_{i\bar{k}, \bar{j}} = \partial h_{i\bar{k}} / \partial \bar{w}^j$ .

The four dimensional space-time is of the form  $\mathbf{R}^2 \times \Sigma$  where  $\Sigma$  is a two-dimensional surface with Euler number

$$\chi(\Sigma) = -\frac{i}{2\pi} \int d^2z \partial \bar{\partial} \rho. \quad (2.7)$$

The energy per unit length of such configurations in complex notation is

$$E = \frac{i}{2} \int d^2z h_{i\bar{j}} \left( \partial w^i \bar{\partial} \bar{w}^j + \partial w^i \partial \bar{w}^j \right), \quad (2.8)$$

and it is easy to verify that it satisfies the BPS bound

$$E \geq \left| \frac{i}{2} \int d^2z h_{i\bar{j}} \left( \partial w^i \bar{\partial} \bar{w}^j - \partial w^i \partial \bar{w}^j \right) \right|. \quad (2.9)$$

The BPS saturated states are then holomorphic (anti-holomorphic) functions  $w^i = w^i(z)$  ( $w^i = w^i(\bar{z})$ ) with energy per unit length

$$E = \frac{i}{2} \int d^2 z h_{i\bar{j}} \partial w^i \bar{\partial} \bar{w}^j. \quad (2.10)$$

By recalling that  $h_{i\bar{j}} = \partial_i \partial_{\bar{j}} K$  we may express the energy in terms of the Kähler potential  $K$  as

$$E = \frac{i}{2} \int_{w(\Sigma)} \partial \bar{\partial} K(w, \bar{w}), \quad (2.11)$$

where  $w(\Sigma)$  is the image of  $\Sigma$  in  $\mathcal{K}$  and here  $(\partial, \bar{\partial})$  are Dolbeault operators. Although  $E$  looks to be a total derivative and thus it should be zero in the compactified  $z$ -plane, it is not, since the Kähler potential  $K$  is not a globally defined quantity. We will verify this later when we will explicitly calculate the integral in eq.(2.11).

The equations for  $w^i, \bar{w}^i$  are automatically satisfied if the BPS condition is fulfilled. One may verify that holomorphic or antiholomorphic  $w^i$  indeed solves eq.(2.6). Thus, only the Einstein equations eq.(2.5) remain to be solved. They turn out to be the single equation

$$\partial \bar{\partial} \rho = -h_{i\bar{j}} \partial w^i \bar{\partial} \bar{w}^j, \quad (2.12)$$

for the conformal factor  $\rho(z, \bar{z})$ . In terms of the Kähler potential  $K$ , eq.(2.12) is written as

$$\partial \bar{\partial} \rho = -\partial \bar{\partial} K. \quad (2.13)$$

The solutions to eq.(2.13) is expressed in terms of an arbitrary holomorphic function  $F(z)$  as

$$\rho(z, \bar{z}) = -K(w, \bar{w}) + F(z) + F(\bar{z}). \quad (2.14)$$

Thus, the metric for the static cylindrically symmetric space-time turns out to be

$$ds^2 = -dt^2 + dx_3^2 + e^{-K(w, \bar{w})} |h(w)|^2 dz d\bar{z}, \quad (2.15)$$

where, by taking into account the holomorphicity of the field  $w$  we have written  $h(w) = \exp F(z)$ . The holomorphic function  $F(z)$  or  $h(w)$  can be specified by demanding non-degenerate metric. Moreover, by comparing eqs.(2.7,2.10,2.12), the energy per unit length is expressed in terms of the Euler number of  $\Sigma$  as

$$E = 2\pi \chi(\Sigma). \quad (2.16)$$

As an explicit example, let us consider the case of an  $SU(2)/U(1)$   $\sigma$ -model [25] with Kähler potential, in projective coordinates  $(w, \bar{w})$ ,  $K = 2n \log(1 + w\bar{w})$  (the factor  $2n$  in front is necessary for the scalar manifold to be a Hodge manifold [26]). Then the metric (2.15) turns out to be

$$ds^2 = -dt^2 + dx_3^2 + \frac{|h(w)|^2}{(1 + w\bar{w})^{2n}} dz d\bar{z}. \quad (2.17)$$

Finite energy solutions are provided by the instanton configurations

$$w(z) = \sum_{i=1}^N \frac{z - a_i}{z - b_i}, \quad (2.18)$$

and thus, the metric vanishes as  $|h(w)|^2 \prod_{i=1}^N |z - b_i|^{4n}$ . The condition of a nowhere vanishing metric leads to the choice  $h(w) = 1 / \prod_i (z - b_i)^{2n}$  so that we get

$$ds^2 = -dt^2 + dx_3^2 + \frac{1}{(1 + w\bar{w})^2 \prod_{i=1}^N |z - b_i|^{4n}} dz d\bar{z}. \quad (2.19)$$

The energy per unit length of these configurations is  $E = 2\pi nN$ . At infinity ( $|z| \rightarrow \infty$ ), the metric of the transverse space goes like  $e^\rho \sim 1/|z|^{4nN}$  and thus there exists a deficit angle  $\delta = 2\pi nN$ . We recall that a deficit angle of  $2\pi$  corresponds to an asymptotically cylindrical space, a deficit angle greater than  $2\pi$  to a conical space with infinity in finite distance while a deficit angle of  $4\pi$  corresponds to  $\mathbf{CP}^1$ . Put differently, a deficit angle  $2\pi$  ( $4\pi$ ) corresponds to a surface with Euler number  $\chi = 1$  ( $\chi = 2$ ). Thus, if  $n = 1$ , two-string configurations compactify the transverse space on  $\mathbf{CP}^1$  while if  $n = 2$ , one string is enough to close it up.

## 2.2 Stringy cosmic strings

We will describe here the stringy cosmic string solution of Greene et al. [18] which is the prototype of solutions we are going to construct. Let us consider the  $SL(2, \mathbf{R})/U(1)$   $\sigma$ -model coupled to gravity in 4-dimensions which is obtained after dimensional reduction of the 6-dimensional Einstein gravity on a torus  $T^2$ . If one fixes the volume of  $T^2$  to some constant value, the only massless moduli is then a complex scalar field which is the complex structure modulus  $\tau$  of the torus. The target space  $\mathcal{M} = SL(2, \mathbf{R})/U(1)$  is the upper half

plane  $\mathcal{H}_1$  which is Kähler with Kähler potential  $K = -\log \tau_2$  ( $\tau_2 = \text{Im} \tau > 0$ ). The bosonic part of the low energy effective action is

$$I = \int d^4x \sqrt{-g} \left( \frac{1}{2} R + \frac{\partial_\mu \tau \partial^\mu \bar{\tau}}{(\tau - \bar{\tau})^2} \right), \quad (2.20)$$

and it is invariant under the global  $SL(2, \mathbf{R})$  transformations

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{R}). \quad (2.21)$$

By identifying the complex scalar  $\tau$  with the complex structure moduli of the internal torus, the theory is invariant under the modular group  $PSL(2, \mathbf{Z}) = SL(2, \mathbf{Z})/\mathbf{Z}_2$  since  $PSL(2, \mathbf{Z})$  transformations of  $\tau$  give back the same torus. The generators of the modular group are the transformations  $\tau \rightarrow \tau + 1$  and  $\tau \rightarrow -1/\tau$ . The equations of motion are now

$$R_{\mu\nu} = -\frac{\partial_\mu \tau \partial_\nu \bar{\tau}}{(\tau - \bar{\tau})^2} - \frac{\partial_\nu \tau \partial_\mu \bar{\tau}}{(\tau - \bar{\tau})^2}, \quad (2.22)$$

$$0 = \frac{1}{\sqrt{-g}} \partial_\mu (g^{\mu\nu} \sqrt{-g} \partial_\nu \tau) + 2 \frac{\partial_\mu \tau \partial^\mu \tau}{\tau - \bar{\tau}}, \quad (2.23)$$

and the stringy cosmic string will be described by a metric of the form eq.(2.3). The equation for  $\tau = \tau(z, \bar{z})$  turns out then to be

$$\partial \bar{\partial} \tau + 2 \frac{\partial \tau \bar{\partial} \tau}{\tau - \bar{\tau}} = 0, \quad (2.24)$$

and it is solved for holomorphic or antiholomorphic field. We will assume that  $\tau = \tau(z)$  in the following. The energy per unit length according to eq.(2.11), is then

$$E = -\frac{i}{2} \int \partial \bar{\partial} \ln \tau_2, \quad (2.25)$$

and it diverges. In order to find finite energy solutions one has to restrict  $\tau$  to the fundamental domain of  $PSL(2, \mathbf{Z})$  [18]. Then,  $\tau$  has discontinuous jumps done by the  $PSL(2, \mathbf{Z})$  transformations  $\tau \rightarrow \tau + 1$  as we go around the string. These jumps and the holomorphicity require that near the location of the string

$$\tau \simeq \frac{1}{2\pi i} \ln z. \quad (2.26)$$

The energy in this case is indeed finite and it turns out to be proportional to the volume of the fundamental domain  $\mathcal{F}_1$ ,

$$E = \frac{\pi}{6} n, \quad (2.27)$$



where  $n$  is the number of times the  $z$ -plane covers  $\mathcal{F}_1$ .

Since the fundamental domain of  $SL(2, \mathbf{Z})$  is mapped to the complex sphere in the  $j$ -plane through the modular  $j$ -function, we may express the solution for  $\tau$  as the pull-back of  $j(\tau)$ . Thus we may write

$$j(\tau) = \frac{P(z)}{Q(z)}, \quad (2.28)$$

where  $P(z), Q(z)$  are polynomials of degree  $p$  and  $q$ , respectively. If  $p \leq q$ ,  $j$  approaches a constant value as  $|z| \rightarrow \infty$  and  $n = q$  in this case. There exist  $q$  points at which  $Q(z)$  has zeroes and these points may be considered as the locations of the string cores. On the other hand, if  $p > q$ , the solution diverges at  $|z| \rightarrow \infty$  and  $n = p$  now.

Turning to the Einstein equations eqs.(2.22), only the (00) equation is not automatically satisfied and it is written as

$$\partial\bar{\partial}\rho = \frac{\partial\tau\bar{\partial}\bar{\tau}}{(\tau - \bar{\tau})^2}. \quad (2.29)$$

By recalling the general discussion of the previous section or by an explicit calculation, one may easily verify that eq.(2.29) is solved by

$$\rho = \tau_2 |h(\tau)|^2, \quad (2.30)$$

where  $h(\tau)$  is an arbitrary holomorphic function. The latter can be specified by demanding non-degenerate metric as well as modular invariance. These two conditions give the supersymmetric solution

$$e^\rho = \tau_2 \eta(\tau)^2 \bar{\eta}(\bar{\tau})^2 \left| \prod_{i=1}^n (z - z_i)^{-1/12} \right|^2, \quad (2.31)$$

where  $\eta(\tau) = q^{1/24} \prod_{r>0} (1 - q^r)$  is the Dedekind's  $\eta$ -function ( $q = e^{2\pi i \tau}$ ). The asymptotic form of the space-time metric is then

$$ds^2 \sim -dt^2 + dx^3 + (z\bar{z})^{-n/12} dz d\bar{z}, \quad (2.32)$$

and one recognizes a deficit angle  $\delta = \pi n/6$ . With  $n = 12$  strings the deficit angle becomes  $\delta = 2\pi$  and the transverse space is asymptotically a cylinder while  $n = 24$  strings produce a deficit angle  $\delta = 4\pi$  and the transverse space is a compact  $\mathbf{CP}^1$ .

Before closing this section, let us also note that this solution is also the prototype of the seven-brane solution [27]. In the latter case, the modulus  $\tau$  corresponds to the ten-dimensional axion-dilaton field of type IIB theory while the  $PSL(2, \mathbf{Z})$  symmetry to strong–weak coupling duality. The seven-branes break half of the space-time supersymmetries and 24 of them compactify the transverse space on  $\mathbf{CP}^1$ . This configuration may then be viewed as a consistent type IIB vacuum. If in addition, one identifies the axion-dilaton field with the complex structure modulus  $\tau$  of a torus compactification of a twelve-dimensional theory [28], the 24 seven-brane configuration of type IIB theory corresponds to a  $K3$  compactification of the twelve dimensional F–theory [9].

### 2.3 Four-dimensional N=4 stringy cosmic strings

The effective action for N=4 supergravity can be obtained by dimensional reduction on a six torus of the ten-dimensional N=1 supergravity coupled to N=1 super Yang–Mills theory [29]. If one restrict himself in the  $U(1)^{16}$  part of the gauge group in ten dimensions, which is also that part that will give rise to massless moduli, the four-dimensional action is [4]

$$I_H = \int d^4x e^{-2\phi} \left( R + 4\partial_\mu \phi \partial^\mu \phi - \frac{1}{12} H_{\mu\nu\kappa} H^{\mu\nu\kappa} - \frac{1}{4} F_{\mu\nu}^I (LML)_{IJ} F^{J\mu\nu} + \frac{1}{8} \text{Tr}(\partial_\mu ML \partial^\mu ML) \right), \quad (2.33)$$

where  $F_{\mu\nu}^I = \partial_\mu A_\nu^I - \partial_\nu A_\mu^I$ , ( $I = 1, \dots, 28$ ),  $H_{\mu\nu\kappa} = \partial_\mu B_{\nu\kappa} + 2A_\mu^I L_{IJ} F_{\nu\kappa}^J + \text{cyclic perm.}$ , and  $M$  is a  $(28 \times 28)$ –matrix which satisfies

$$MLM^T = L, \quad M^T = M, \quad L = \begin{pmatrix} 0 & I_6 & 0 \\ I_6 & 0 & 0 \\ 0 & 0 & -I_{16} \end{pmatrix}. \quad (2.34)$$

The entries of  $M$  are expressed in terms of the scalars of the theory which parametrize the coset  $O(6, 22)/O(6) \times O(22)$  [30]. The effective action is invariant under the  $O(6, 22)$  transformations

$$M \rightarrow \Omega M \Omega^T, \quad A_\mu \rightarrow \Omega A_\mu, \quad (2.35)$$

which leave all other fields invariant and where  $\Omega$  is an  $O(6, 20)$  matrix satisfying  $\Omega^T L \Omega = L$ .

We will consider here only an  $O(2,2)/O(2) \times O(2)$  subspace of the full moduli space which can be obtained from six dimensions as follows. By toroidal compactification of the ten-dimensional N=1 supergravity we get N=2 supegravity in six dimensions with moduli space  $O(4,20)/O(4) \times O(20)$ . Further compactification on a two torus will give N=4 in four-dimensions with moduli space

$$\frac{O(2,2)}{O(2) \times O(2)} \times \frac{O(4,20)}{O(4) \times O(20)}, \quad (2.36)$$

if there are no components of the six-dimensional gauge fields along the two torus. We will consider only the first factor of (2.36) and we will fix all other moduli to some constant value. Combined with the dilaton S-field, we will deal in the following with

$$\mathcal{M} = \frac{SL(2, \mathbf{R})}{U(1)} \times \frac{O(2,2)}{O(2) \times O(2)}, \quad (2.37)$$

and the only non-vanishing scalars will be the dilaton S and the Kähler and complex structure moduli of the torus T and U, respectively. They are defined as

$$\begin{aligned} S &= \alpha + ie^{-2\phi}, \\ T &= B_{45} + i\sqrt{\det G_{mn}}, \\ U &= \frac{G_{45}}{G_{55}} + i\frac{\sqrt{\det G_{mn}}}{G_{55}}, \end{aligned} \quad (2.38)$$

where  $G_{mn}(m, n = 4, 5)$  and  $B_{45}$  are the metric and the component of the antisymmetric tensor on the torus and  $\alpha, \phi$  are the axion and the dilaton, respectively. The bosonic part of the action is then

$$I = \int d^4x \sqrt{-g} \left( \frac{1}{2}R - \frac{\partial_\mu S \partial^\mu \bar{S}}{(S - \bar{S})^2} - \frac{\partial_\mu T \partial^\mu \bar{T}}{(T - \bar{T})^2} - \frac{\partial_\mu U \partial^\mu \bar{U}}{(U - \bar{U})^2} \right). \quad (2.39)$$

The T-duality group is  $SL(2, \mathbf{Z})_T \times SL(2, \mathbf{Z})_U$  and in addition, the theory is believed to be also invariant under the S-duality group  $SL(2, \mathbf{Z})_S$ . These discrete groups act on the fields as

$$S \rightarrow \frac{aS + b}{cS + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{R})_S, \quad (2.40)$$

and similarly for  $T, U$ . There is also another discrete symmetry, the string/string/string triality which interchanges  $S \leftrightarrow T \leftrightarrow U$  [31]. Although part of the triality is realized on-shell, here is manifest at the level of the action since we have turned off all gauge fields.

Solitonic string solutions of the form eq.(2.3) can be constructed by recalling that the prepotential is  $F = STU$  and the Kähler potential  $K = -\log(S_2 T_2 U_2)$  ( $S_2, T_2, U_2$  are the imaginary parts of the  $S, T, U$  moduli). Then, from eq.(2.15) it follows that the metric is given by

$$ds^2 = -dt^2 + dx_3^2 + S_2 T_2 U_2 |f(z)|^2 dz d\bar{z}. \quad (2.41)$$

The moduli fields  $S, T, U$  are holomorphic and, as before, finiteness of the energy is achieved by restricting them on the fundamental domains of  $SL(2, \mathbf{Z})_S$ ,  $SL(2, \mathbf{Z})_T$  and  $SL(2, \mathbf{Z})_U$ , respectively. In this case, the solution may be expressed as the pull-backs of  $j(S)$ ,  $j(T)$  and  $j(U)$  and  $S, T, U$  will be given by

$$S \sim \frac{1}{2\pi i} \log(z - z_k), \quad (2.42)$$

$$T \sim \frac{1}{2\pi i} \log(z - z_i), \quad (2.43)$$

$$U \sim \frac{1}{2\pi i} \log(z - z_j), \quad (2.44)$$

near the core of the strings. The conditions of modular invariance and non-degeneracy of the metric give now the solution

$$ds^2 = -dt^2 + dx_3^2 + S_2 T_2 U_2 |\eta(S)\eta(T)\eta(U)|^4 \left| \prod_{i=1}^{n_S} \prod_{j=1}^{n_T} \prod_{k=1}^{n_U} (z - z_i)(z - z_j)(z - z_k) \right|^{-1/6} dz d\bar{z} \quad (2.45)$$

where  $n_S, n_T, n_U$  are the number of strings carrying  $S, T$  and  $U$  charge, respectively. The total energy is  $E = (n_S + n_T + n_U)\pi/6$ . Finally, string/string/string triality [31] requires  $n_S = n_T = n_U = n$  while from the asymptotic behaviour of the metric it turns out that  $n = 8$  in order the transverse space to be  $\mathbf{CP}^1$ .

The form of the metric (2.45) indicates that there exist eight S-strings, strings which carry S-charge, eight T-strings and eight U-strings. This configuration compactifies the transverse space on  $\mathbf{CP}^1$ . On the other hand, string/string/string triality allows also for STU-strings, that is strings which carry both S,T and U charges. The transverse space metric for these strings is

$$ds_\perp^2 = S_2 T_2 U_2 |\eta(S)\eta(T)\eta(U)|^4 \left| \prod_{i=1}^8 (z - z_i)^{-1/4} \right|^2 dz d\bar{z}, \quad (2.46)$$

and we will see that in this case the location of the string cores may be at orbifold singularities. From eq.(2.46) it follows that around the eight points  $z_i$  there exist a deficit angle of  $\pi/2$  and it is clear that these points cannot be thought as orbifold singularities. The deficit angle of a fixed point of order  $n$  is  $2\pi(n-1)/n$ . Let us assume that the eight points in eq.(2.46) coalesced into three points of order three, three and two, i.e.,

$$ds_{\perp} = S_2 T_2 U_2 |\eta(S)\eta(T)\eta(U)|^4 \left| (z - z_1)^{-3/4} (z - z_2)^{-3/4} (z - z_3)^{-1/2} \right|^2 dz d\bar{z}. \quad (2.47)$$

Around the points  $z_1, z_2, z_3$  there exist then a deficit angle of  $3\pi/2, 3\pi/2$  and  $\pi/2$ , respectively. This means that the transverse space has been turned into a  $T^2/Z_4$  orbifold. If the eight points coalesced into four points of order two each, the metric (2.46) turns out be

$$ds_{\perp} = S_2 T_2 U_2 |\eta(S)\eta(T)\eta(U)|^4 \left| \prod_{i=1}^4 (z - z_i)^{-1/2} \right|^2 dz d\bar{z}. \quad (2.48)$$

There exist now a deficit angle of  $\pi/2$  around each of the four points and thus the transverse space has been turned into a  $T^2/Z_2$  orbifold. The above configurations correspond to special points in the moduli space with constant fields as has been discussed in [32] for the seven-branes of type IIB.

### 3. N=2 heterotic stringy cosmic strings

Solitonic string solutions exist as we will see in the N=2 heterotic theory as well. We will explicitly construct here these heterotic N=2 four-dimensional solutions in the rank three, four and five models. These models can be obtained by reduction of N=1 supergravity coupled to N=1 super Yang–Mills in six dimensions on a torus. There are no Wilson line moduli for the rank three and four models while for the rank five model there exists a single Wilson line moduli.

#### 3.1 The rank three $S - T$ model

We will consider first the rank three model. In this case, the vector multiplets contain the dilaton  $S$  and the  $T$  modulus which parametrize the coset  $O(2,1)/O(2)$ . The classical T-duality group is  $SL(2, \mathbf{Z})$ . At a generic point of the  $T$ -moduli space the gauge group is

$U(1)^3$  while at  $T = i$ , two extra vector multiplets become massless leading to an enhanced gauge group  $U(1)^2 \times SU(2)$ . One should expect then that the solitonic string solutions will be given by an expression similar to eq.(2.45) with  $U$  constant. However, one should take into account that the classical moduli space for the N=2 case receives quantum corrections. Moreover, the S–duality group  $SL(2, \mathbf{Z})_S$  is not expected to be a symmetry of the full quantum theory.

The classical prepotential and the Kähler potential for the model we consider are

$$\mathcal{F}^{(0)} = \frac{1}{2}ST, \quad (3.1)$$

$$K^{(0)} = -\log(S - \bar{S}) - 2\log(T - \bar{T})^2. \quad (3.2)$$

If the above expressions were exact one might proceed in the construction of the solitonic strings as in sect. 2.3. In the N=2 case, however, the classical prepotential and consequently, the Kähler potential, receives quantum corrections, both perturbatively and non-perturbatively. Here we will consider only perturbative corrections and the solution we will construct will be perturbatively exact. Due to the N=2 non-renormalization theorems, there exist only one-loop corrections to the classical prepotential, denoted by  $h(T, U)$ , as well as to the Kähler potential. In fact, the classical expressions (3.1,3.2) are modified by quantum corrections and they turn out to be

$$\mathcal{F} = \frac{1}{2}ST + h(T) + \dots, \quad (3.3)$$

$$K = -\log(S - \bar{S} - V_{GS}) - 2\log(T - \bar{T}). \quad (3.4)$$

where the Green–Schwarz term  $V_{GS}$  is

$$V_{GB} = 4\frac{h - \bar{h}}{(T - \bar{T})^2} - 2\frac{\partial_T h + \partial_{\bar{T}} \bar{h}}{T - \bar{T}}, \quad (3.5)$$

and the dots in eq.(3.3) refer to exponentially suppressed non-perturbative corrections. In addition, the requirement that the  $SL(2, \mathbf{Z})$  T–duality transformation  $T \rightarrow \frac{aT+b}{cT+d}$  is a Kähler transformation, implies that both the dilaton  $S$  and  $h(T)$  transforms as

$$\begin{aligned} h(T) &\rightarrow \frac{h(T)}{(cT+d)^4} + \frac{B(T)}{(cT+d)^4}, \\ S &\rightarrow S - \frac{1}{3}\partial_T^2 B + 2c\frac{\partial_T h + \partial_T B}{cT+d} - 4c^2\frac{h+B}{(cT+d)^2} + const. \end{aligned} \quad (3.6)$$

where  $B(T)$  is at most a quartic polynomial in  $T$  with real coefficients [16],[17]. Then, the Kähler potential in eq.(3.4) transforms under  $SL(2, \mathbf{Z})$  as

$$K \rightarrow K + 2 \log(cT + d) + 2 \log(c\bar{T} + d), \quad (3.7)$$

which is indeed a Kähler transformation.

As long as S–duality is not expected to be a symmetry of the  $N = 2$  string, solitonic string solutions can be constructed by fixing the dilaton S–field to some constant value and employing the  $SL(2, \mathbf{Z})$  T–duality group for the T modulus. Then, following the general discussion of sect. 2.1, we find that the metric of the N=2 solitonic string of the rank three model is

$$ds^2 = -dt^2 + dx_3^2 + (S_2 - 2iV_{GS})T_2^2 |\eta(T)|^8 \left| \prod_{i=1}^n (z - z_i)^{-1/6} \right|^2 dz d\bar{z}, \quad (3.8)$$

where  $\eta(T)$  is the Dedekind’s  $\eta$ –function. Near the string core we expect that

$$T \sim \frac{1}{2\pi i} \log(z - z_i), \quad (3.9)$$

and thus,  $q \sim (z - z_i)$ . One may then verify that the metric is modular invariant and has no zeroes in the complex plane. The energy on the other hand is finite and for a single string configuration it is twice the energy of the corresponding N=4 stringy cosmic string, i.e.,

$$E = n \frac{\pi}{3}.$$

As a result, twelve strings compactify the transverse space on  $\mathbf{CP}^1$  providing a consistent vacuum configuration.

### 3.2 The rank four $S - T - U$ model

Let us now consider the rank four N=2 heterotic  $S - T - U$  model. The moduli space in this case is

$$\frac{SL(2, \mathbf{R})}{U(1)} \times \frac{O(2, 2)}{O(2) \times O(2)},$$

where the first factor is the moduli space of the S–dilaton and the second factor for the  $T$  and  $U$  moduli of the internal torus. The classical T–duality group is in this case  $O(2, 2, \mathbf{Z}) =$

$SL(2, \mathbf{Z})_T \times SL(2, \mathbf{Z})_U$  modulo  $T \leftrightarrow U$  interchange. The classical prepotential and the Kähler potential for the  $S - T - U$  model are

$$\mathcal{F}^{(0)} = -STU, \quad (3.10)$$

$$K^{(0)} = -\log(S - \bar{S}) - \log((T - \bar{T})(U - \bar{U})), \quad (3.11)$$

where  $S, T, U$  have been defined in eq.(2.38). In the N=4 case considered in sect. 2.3,  $\mathcal{F}^{(0)}$  and  $K^{(0)}$  were exact and together with S- and T-duality employed in the construction of the N=4 stringy cosmic string. Here however, as in the rank three model, quantum corrections modify both the prepotential and the Kähler potential which turn out to be

$$\mathcal{F} = \mathcal{F}^{(0)} + h(T, U) + \dots, \quad (3.12)$$

$$K = -\log(S - \bar{S} + V_{GS}) - \log((T - \bar{T})(U - \bar{U})), \quad (3.13)$$

where the Green-Schwarz terms is

$$V_{GS} = -2 \frac{h - \bar{h}}{(T - \bar{T})(U - \bar{U})} + \frac{\partial_T h + \partial_{\bar{T}} \bar{h}}{(U - \bar{U})} - \frac{\partial_U h + \partial_{\bar{U}} \bar{h}}{(T - \bar{T})}. \quad (3.14)$$

Proceeding as before we find that the N=2 stringy cosmic string for the  $S - T - U$  model has metric

$$ds^2 = -dt^2 + dx_3^2 + (S_2 - 2iV_{GS})T_2U_2|\eta(T)\eta(U)|^4 \left| \prod_{i=1}^{n_T} \prod_{j=1}^{n_U} (z - z_i)(z - z_j) \right|^{-1/6} dzd\bar{z}, \quad (3.15)$$

where  $n_T, n_U$  are the number of times the  $z$ -plane covers the fundamental domains of  $SL(2, \mathbf{Z})_T$  and  $SL(2, \mathbf{Z})_U$ , respectively. The  $T \leftrightarrow U$  exchange symmetry is broken by quantum corrections and the numbers  $n_T, n_U$  cannot be related any more as in 2.3. The energy turns out to be

$$E = \frac{\pi}{6}(n_T + n_U), \quad (3.16)$$

and the regularity of the solution requires  $n_T + n_U = 24$ .

### 3.3 The rank five $S - T - U - V$ model

An interesting case of string-string dualities is provided by the heterotic string in D=10 compactified on  $K3 \times T^2$  which is related to a type II string compactified on a appropriate



Calabi-Yau three-fold. There exist successful tests of this duality for models with small number of vector multiplets [22],[23]. In particular, with  $N_V = 4$  massless Abelian vector multiplets one is dealing with the S-dilaton and the complex fields T and U (besides the graviphoton) where T and U are the torus moduli. Solitonic string solutions of this model has been discussed in the previous chapter. Here, we will consider the case where additional massless Wilson line moduli exist. In the presence of p non-vanishing Wilson lines, the classical vector multiplet moduli space of N=2 string compactification turns out to be locally the special Kähler manifold [34], [35]

$$\frac{SL(2, \mathbf{R})}{U(1)} \times \frac{O(2, 2+p)}{O(2) \times O(2+p)},$$

where again the first factor is the S-field moduli space. The classical T-duality group is  $O(2, 2+p; \mathbf{Z})$ . For the special p=1 case we will discuss here, there exist a single Wilson line V and the moduli space is

$$\frac{SL(2, \mathbf{R})}{U(1)} \times \frac{O(2, 3)}{O(2) \times O(2+p)}.$$

The T-duality group is  $O(2, 3; \mathbf{Z})$  which is isomorphic to  $Sp(4, \mathbf{Z})$ . A short account of its properties are presented in the appendix.

The loop-corrected prepotential and Kähler potential for the  $S - T - U - V$  model are

$$\mathcal{F} = -S(TU - V^2) + h(T, U, V), \quad (3.17)$$

$$K = -\log(S - \bar{S} + V_{GS}) - \log\left((T - \bar{T})(U - \bar{U}) - (V - \bar{V})^2\right), \quad (3.18)$$

where  $h(T, U, V)$  is the one-loop prepotential and the Green-Schwarz term  $V_{GS}$  is expressed in terms of  $h$  as

$$V_{GS} = \frac{(T - \bar{T})(h_T + \bar{h}_{\bar{T}}) + (U - \bar{U})(h_U + \bar{h}_{\bar{U}}) + (V - \bar{V})(h_V + \bar{h}_{\bar{V}})}{\left((T - \bar{T})(U - \bar{U}) - (V - \bar{V})^2\right)} - \frac{2(h - \bar{h})}{\left((T - \bar{T})(U - \bar{U}) - (V - \bar{V})^2\right)^2}, \quad (3.19)$$

where e.g.  $h_T = \partial_T h$ . The T-duality transformation  $\tau \rightarrow (a\tau + b)(c\tau + d)^{-1}$  is a Kähler transformation and the Kähler potential transforms as

$$K \rightarrow K + \log(\det(c\tau + d)) + \log(\det(c\bar{\tau} + d)), \quad (3.20)$$

where  $\tau$  is defined in eq.(A.2).

Finite energy solitonic string solutions for the  $S - T - U - V$  model may be constructed by fixing the dilaton to some constant value and employing the  $Sp(4, \mathbf{Z})$  T-duality group. We will allow again the  $\tau$ -field to have discontinues jumps as we go around the string. Then, near the core of the string, we will have

$$T \sim \frac{1}{2\pi i} \log(z - z_i), \quad U \sim \frac{1}{2\pi i} \log(z - z_j) \quad V \sim \frac{1}{2\pi i} \log(z - z_k). \quad (3.21)$$

According to eq.(2.15), the metric for the  $S - T - U - V$  model takes the form

$$ds^2 = -dt^2 - dx_3^2 + (S_2 - 2iV_{GS})(T_2U_2 - V_2^2)|f(T, U, V)|^2 dz d\bar{z}. \quad (3.22)$$

The function  $f(T, U, V)$  will be determined by demanding modular invariance and no degenerate metric. It follows from eq.(3.22) that in order to achieve modular invariance  $f(T, U, V)$  must contain a factor which transforms as a modular form of weight +1 and has no zeroes in the fundamental domain  $\mathcal{F}_2$ . The unique form with this properties is the twelfth root of the cusp form  $\Psi_{12}$ . However, although the latter has no zeroes in  $\mathcal{F}_2$  it might have zeroes in the  $z$ -plane at the locations of the string core. There we have

$$q = \exp(2\pi iT) \sim (z - z_i), \quad s = \exp(2\pi iU) \sim (z - z_j), \quad r = \exp(2\pi iV) \sim (z - z_k)$$

as it follows from eq.(3.21). Although a product expression for  $\Psi_{12}$  is lacking [39], we now that [36]

$$\Psi_{12} = qs + \dots, \quad (3.23)$$

which may also be seen from the degeneration limit  $V \rightarrow 0$  at which

$$\Psi_{12} \rightarrow \Delta(q)\Delta(s). \quad (3.24)$$

Thus, the conditions of modular invariance and non-degeneracy of the metric give

$$ds^2 = -dt^2 - dx_3^2 + (S_2 - 2iV_{GS})(T_2U_2 - V_2^2) |\Psi_{12}|^{1/6} \left| \prod_{i=1}^{n_T} \prod_{j=1}^{n_U} |(z - z_i)(z - z_j)| \right|^{-1/6} dz d\bar{z} \quad (3.25)$$

In the degeneration limit one recovers the solution of the  $S - T - U$  model eq.(3.15). The solution for the  $T, U, V$  moduli will be given as the pull-backs of the modular invariant functions  $x_1, x_2$  and  $x_3$  defined in the Appendix. The energy finally is indeed finite

$$E = \frac{\pi}{6}(n_T + n_U), \quad (3.26)$$

and regularity demands  $n_T + n_U = 24$ .

## 4. Conclusions

We have constructed here solitonic string solutions of four-dimensional  $N=4$  and  $N=2$  heterotic theories. In the  $N=4$  case, we have considered the dilaton with two additional moduli. By employing the S- and T-duality groups as well as string/string/string triality we were able to explicitly find string-like configurations with finite energy per unit length. Regular solutions are provided then by twenty-four strings since in this case the transverse space is compactified on  $\mathbf{CP}^1$ . This is closely related with the fact that there exist elliptically fibered manifolds with base space  $\mathbf{CP}^1$  as for example K3 surfaces or Calabi–Yau three-folds which admit two elliptic fibrations. Then, the singularities may be resolved in a Ricci-flat way, consistently with supersymmetry [18].

The solitonic strings of the  $N=4$  case may also be seen from a type II point of view. One may compactify type II theory on a  $(T^2)^3$  which gives  $N=8$  supersymmetry in four dimensions. By fixing the Kähler structure moduli of the tori to some constant value, the only moduli which will appear in the four-dimensional effective theory will be three complex scalars corresponding to the complex structures of the internal tori. In this case, one may express the solutions in terms of elliptic curves as has been done in the seven-brane solution in type IIB theory in ten dimensions [9].

We have also constructed solitonic string solutions of  $N=2$  heterotic models of rank three, four and five. In the first two cases there are no Wilson line moduli and the solutions were found by employing the T-duality groups  $SL(2, \mathbf{Z})$  and  $SL(2, \mathbf{Z}) \times SL(2, \mathbf{Z})$ . We kept the dilaton constant as explained already since otherwise the construction of finite energy solutions would not be possible. We have also consider the case where a single Wilson line moduli is present. In this case the T-duality group is  $O(3, 2; \mathbf{Z})$  which is isomorphic to  $Sp(4, \mathbf{Z})$  and the solitonic string solution was expressed in terms of modular forms of the latter. It should be noted that in the general case where  $p$  Wilson line moduli are turned on, the T-duality group is  $O(2+p, 2; \mathbf{Z})$  and we need to know its modular forms with no zeroes in  $O(2+p, 2; \mathbf{Z}) \backslash O(2+p, 2; \mathbf{R}) / O(2+p) \times O(2)$ . These forms are not explicitly known and a way to construct them may probably be based on [43] where automorphic forms of  $O(2+p, 2; \mathbf{R})$  with well controlled pole structure have been introduced.

Finally, it should be noted that a single solitonic string is not a consistent solution. One

has to consider multi-string configurations as consistent string backgrounds. In this case, the transverse space is compact and at a generic point of the moduli space, it is a sphere. At special points, however, it turns into an orbifold. The location of the string cores are then at the fixed points of that orbifold.

## Appendix

Here we review some properties of  $Sp(4, \mathbf{Z})$  [36]–[42]. The group  $Sp(4, \mathbf{Z})$  is subgroup of  $Sp(4, \mathbf{R})$  and consists of all integral  $4 \times 4$  matrices

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

such that  $M^t J M = J$  where  $a, b, c, d$  are integral  $2 \times 2$  matrices and

$$J = \begin{pmatrix} 0 & \mathbf{1}_{2 \times 2} \\ -\mathbf{1}_{2 \times 2} & 0 \end{pmatrix}.$$

The standard action of  $Sp(4, \mathbf{Z})$  on the Siegel upper half space  $\mathcal{H}_2 = O(2, 3)/O(2) \times O(3)$  is given by

$$\tau \rightarrow (a\tau + b)(c\tau + d)^{-1}, \quad (\text{A.1})$$

where

$$\tau = \begin{pmatrix} T & V \\ V & U \end{pmatrix} \in \mathcal{H}_2, \quad (\text{A.2})$$

with  $Im\tau = (T_2 U_2 - V_2^2) > 0$ <sup>1</sup> Similarly to the  $SL(2, \mathbf{Z})$  group,  $Sp(4, \mathbf{Z})$  is generated by

$$\begin{pmatrix} 0 & \mathbf{1}_{2 \times 2} \\ -\mathbf{1}_{2 \times 2} & 0 \end{pmatrix}, \quad \begin{pmatrix} A & 0 \\ 0 & A^* \end{pmatrix}, \quad \begin{pmatrix} \mathbf{1}_{2 \times 2} & B \\ 0 & \mathbf{1}_{2 \times 2} \end{pmatrix},$$

where  $A \in GL(2, \mathbf{Z})$  with  $A^* = (A^t)^{-1}$  and

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The  $Sp(4, \mathbf{Z})$  fundamental domain  $\mathcal{F}_2$  can be defined by the conditions

<sup>1</sup> $T_1, T_2, U_1, U_2, V_1, V_2$  are the real and imaginary parts of the complex fields T, U and V respectively.

1.  $|T_1| \leq \frac{1}{2}, |U_1| \leq \frac{1}{2}, |V_1| \leq \frac{1}{2},$
2.  $0 \leq |2V_2| \leq T_2 \leq U_2,$
3.  $|\det(c\tau + d)| \geq 1$  for all  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(4, \mathbf{Z}).$

A Siegel modular form  $F$  of weight  $k$  is a holomorphic function on  $\mathcal{H}_2$  with the property

$$F((a\tau + b)(c\tau + d)^{-1}) = \det(c\tau + d)^k F(\tau). \quad (\text{A.3})$$

Such forms appear in string theory in two-loop amplitudes [41] as well as in some recent developments [42]. Examples of such modular forms are provided by the Eisenstein series

$$E_k(\tau) = \sum_{c,d} \frac{1}{\det(c\tau + d)}, \quad (\text{A.4})$$

where the summation is over all inequivalent bottom rows of elements of  $Sp(4, \mathbf{Z})$ . One may easily prove that  $E_k(\tau)$  are modular with weight  $k$  for  $k > 3$ . The Eisenstein series  $E_4, E_6, E_{10}$  and  $E_{12}$  are algebraically independent over  $\mathbf{C}$  and they generate the graded ring of even modular forms. Similarly to the  $SL(2, \mathbf{Z})$  case, there are also cusp forms for the group  $Sp(4, \mathbf{Z})$  which are the forms  $\Psi_{10}$  of weight 10,  $\Psi_{12}$  of weight 12 and  $\Psi_{35}$  of weight 35. In addition, one defines  $\Psi_5 = \Psi_{10}^{1/2}$  and  $\Psi_{30} = \Psi_{35}/\Psi_5$ .

There exist also  $SL(4, \mathbf{Z})$  modular functions, the counterparts of the  $j$ -modular function. There exist three such functions which can be written as

$$x_1 = E_4 \Psi_{10}^2 / \Psi_{12}^2, \quad x_2 = E_6 \Psi_{10}^3 / \Psi_{12}^3, \quad x_3 = \Psi_{10}^6 / \Psi_{12}^5. \quad (\text{A.5})$$

In general the cusp forms have zeroes on rational quadratic divisors  $H_\ell$  of  $\mathcal{H}_2$  with discriminant  $D(\ell) = \beta^2 - 4\delta\epsilon - 4\alpha\gamma$ .  $H_\ell$  is defined as the set

$$H_\ell = \left\{ \begin{pmatrix} T & V \\ V & U \end{pmatrix} \in \mathcal{H}_2 \mid \alpha + \beta T + \gamma V + \delta U + \epsilon(V^2 - TU) = 0 \right\},$$

where  $\ell = (\alpha, \beta, \gamma, \delta, \epsilon) \in \mathbf{Z}^5$  is primitive, i.e their greatest common divisor is one. The divisor  $H_\ell$  exists if  $D(\ell) > 0$  and it determines the Humbert surface  $H_D$  in  $Sp(4, \mathbf{Z}) \backslash \mathcal{H}_2$  which is the union of all divisors of discriminant  $D(\ell)$ . The divisors of  $\Psi_{10}$  and  $\Psi_5$  are the Humbert surface

$$H_1 = \{\tau \in Sp(4, \mathbf{Z}) \mid \tau = \begin{pmatrix} T & 0 \\ 0 & U \end{pmatrix}\},$$

the divisors of  $\Psi_{30}$  are the surface

$$H_4 = \{\tau \in Sp(4, \mathbf{Z}) | \tau = \begin{pmatrix} T & V \\ V & T \end{pmatrix}\},$$

and the divisors of  $\Psi_{35}$  is the union of  $H_1$  and  $H_4$ . On the other hand, the unique cusp form without divisors is  $\Psi_{12}$ .

Finally, let us mention that in the degeneration limit  $V \rightarrow 0$

$$\begin{aligned} E_4 &\rightarrow G_4(T)G_4(U), \\ E_6 &\rightarrow G_6(T)G_6(U), \\ \Psi_5 &\rightarrow 0, \\ \Psi_{12} &\rightarrow \Delta(T)\Delta(U), \\ \Psi_{35} &\rightarrow \Delta(T)^{5/2}\Delta(U)^{5/2}(j(T) - j(U)), \end{aligned} \tag{A.6}$$

where  $\Delta = \eta^{24}$  is the  $SL(2, \mathbf{Z})$  cusp form and  $G_4, G_6$  are the Eisenstein series of  $SL(2, \mathbf{Z})$  of weight four and six, respectively.

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